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Stability of a Liquid-Filled Gyroscope: Inviscid Analysis, Viscous Corrections, and Experiments

JOHN T. FRASIER* AND W. EUGENE SCOTT†
Ballistic Research Laboratories, Aberdeen Proving Ground, Md.

Nomenclature

$a,2c,d$	= radius, cavity length, and central rod radius, respectively
C_j	= constants, $-2/ck_j^2$
F	= amplitude of the moment applied to the gyroscope by the liquid, Eqs. (1) and (5)
g	= acceleration due to gravity
$\mathbf{i}_Z$	= unit vector in positive Z direction
j	= longitudinal mode number; $j = 0, 1, 2 \ldots$
k_j	= $\pi(2j + 1)/2c$
l,m,n	= direction cosines of z' with respect to x, y, z
L,T	= axial and transverse (with respect to axes through 0) moments of inertia of the gyroscope
M_0	= a constant moment coefficient, Eq. (1)
$OXYZ$	= inertial coordinate system, Z vertical
$Oxyz$	= rectangular coordinate system which rotates about the vertical z axis with speed Ω, Fig. 1
$Ox'y'z'$	= rectangular coordinate system attached to the gyroscope rotor; z' is symmetry axis of rotor, Fig. 1
P	= pressure function, defined after Eq. (3)
p,p_0	= total pressure in the liquid and a constant pressure, respectively
p^*	= perturbation pressure in the liquid
$\mathbf{R}$	= position vector in the $OXYZ$ frame
R_{nj}	= residue at the pole τ_{nj} of the right side of Eq. (14)
Re	= $a^2\Omega/\nu$ = Reynolds number
r,r'	= $x^2 + y^2$, and $x'^2 + y'^2$, respectively
$\mathbf{s}(X,Y)$	= position vector in planes normal to OZ
t	= time
$\mathbf{u}(u,v,w)$	= fluid perturbation velocity relative to $OXYZ$
$\mathbf{u}^*,\mathbf{u}^+$	= velocities, see after Eqs. (29) and (31)
$\mathbf{v}(u,v,w)$	= fluid velocity vector in the $Oxyz$ system
X_j,Z_j	= constants
α_I	= yaw growth rate factor, inviscid case
α_V	= yaw growth rate factor, viscous case
β	= gyroscopic stability factor, $4M_0T/L^2\Omega^2$
$\delta a,\delta d,\delta c$	= boundary layer thicknesses
ζ,ξ	= terms defined by Eqs. (19) and (7), respectively
η	= $(s - d)$ = boundary-layer coordinate at the rod-liquid interface
λ	= $\lambda_X + i\lambda_Y$, the (small) complex yaw of the gyroscope with respect to the inertial coordinates $OXYZ$
ν	= kinematic viscosity of the liquid
Ξ_γ	= linear combinations of the Bessel functions J_γ, Y_γ
ρ	= liquid density
τ	= general nondimensional frequency, normalized by Ω
$\tau_{nj},\tau_{NU},\tau_{PR}$	= njth eigenfrequency of the liquid, and the nutational and precessional frequencies of the gyroscope, all nondimensional
$\tau_{nj}^v,\tau_{nj}^*,\delta_{nj}$	= the viscous eigenfrequency, and its real and imaginary parts, respectively
ϕ	= angle in cyclindrical polar coordinates
Ω	= spin of the gyroscope

Subscripts

0,1	= inviscid and boundary-layer flows, respectively
n,j	= radial and longitudinal mode numbers of the liquid oscillations
$(\dot{\ })$	= $d(\ )/dt$; $(\ddot{\ }) = d^2(\ )/dt^2$

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* Chief, Solid Mechanics Branch, Terminal Ballistics Laboratory.
† Physicist, Exterior Ballistics Laboratory.

Introduction

IN the symmetric rotor of a gyroscope, liquid fills an axially aligned, cylindrical annular cavity (Fig. 1) around a coaxial rigid rod. We wish to study the stability of motion of this system. The problem is a companion to the one without the central rigid rod analyzed by Stewartson.[1] The significance of these problems resides in the occurrence of liquid payloads and fuels in gyroscopes and gyroscope-like devices (e.g., satellites, projectiles, missiles) and the need to design against violent, unstable motion that can occur if certain eigenfrequencies τ_{nj} of the rotating liquid are critically close to the nutational frequency τ_{NU} of the gyroscope. Hence, we must know the appropriate stability criterion and the τ_{nj}. Analysis and evaluation of τ_{nj} for specific cavity

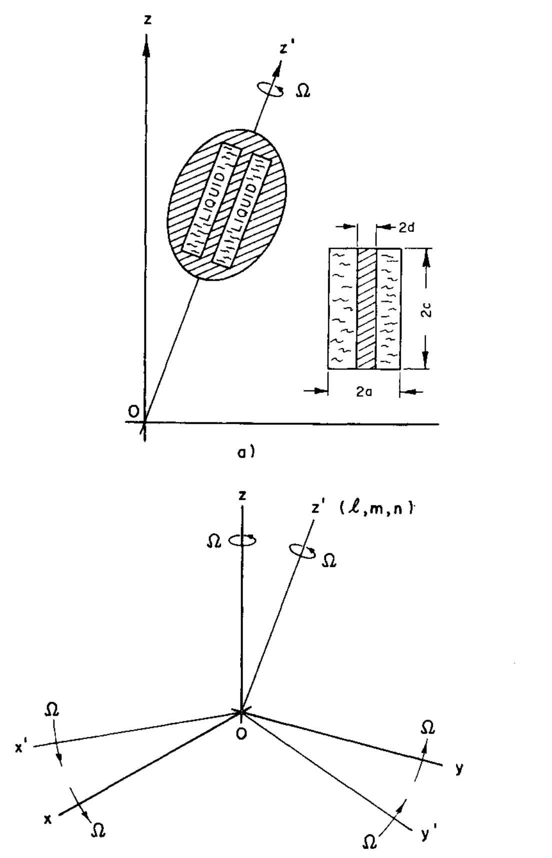

Fig. 1 a) A cross section of the gyroscope in the plane of the $z - z'$ axes. b) The rotating coordinate frames. Rotations depicted are referenced to an inertial frame.

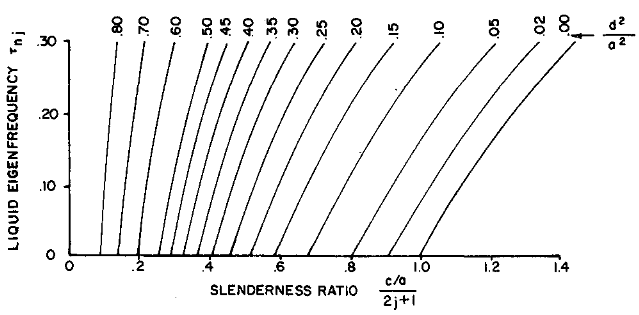

Fig. 2 Eigenfrequencies of the liquid as functions of slenderness ratio and rod size for $n = 1$. See Ref. 3 for a more detailed tabulation.

geometries are difficult, and results exist for relatively few cases.‡ However, for the present problem, analysis was possible, and extensive computations have been made to satisfy practical design requirements confronting the authors. Since the inviscid analysis is parallel in procedure to Stewartson's work,[1] we shall simply introduce the basic assumptions and governing equations, sketch the method of solution, and present the numerically computed results. Then we shall use Wedemeyer's[3] method for taking viscosity into account and compare the theory to some experimental data.[4]

Inviscid Analysis

For the gyroscope in Fig. 1, suppose that 1) the container is completely filled with an inviscid, incompressible liquid; 2) the mass of the liquid is much smaller than that of the metal parts of the rotor (frequently true in practice); 3) Ω is constant and, with the dimensions of the cavity, satisfies the relation $a^2\,\Omega^2 \gg gc$; 4) the primary motion of the liquid is rigid-body spin identical to that of the rotor; and 5) any perturbation of the gyroscope from its sleeping position (spin axis vertical and stationary) and the associated disturbance to the rigid-body spin of the liquid are sufficiently small to justify neglect of terms of the order of squares and products among the direction cosines l,m,n and the perturbed components of fluid velocity u,v,w. With these restrictions we wish to perform a linear (small-disturbance) stability analysis. Under our restrictions the equation governing the perturbed motion is

$$T\ddot{\lambda} - i\Omega L\dot{\lambda} - M_0\lambda = -iF \qquad (1)$$

where $i = (-1)^{1/2}$. The motion of the liquid is governed by

$$\nabla\cdot\mathbf{v} = 0 \qquad \text{(continuity)} \qquad (2)$$

$$\partial\mathbf{v}/\partial t + 2\boldsymbol{\Omega}\times\mathbf{v} = -\nabla P \qquad \text{(balance of momentum)} \qquad (3)$$

where

$$\rho P = p - p_0 - \tfrac{1}{2}\rho\Omega^2(x^2 + y^2 - d^2)$$

Equation (1) is referred to the inertial $OXYZ$ coordinate system; Eqs. (2) and (3), to the rotating $Oxyz$ system. Boundary conditions for the liquid require its normal component of velocity to match that of the cavity boundary, § thus

$$\dot{l}x + \dot{m}y = -w \quad \text{on } z' = h \pm c \qquad (4)$$

$$ux + vy = z(\dot{l}x + \dot{m}y) \text{ on } r' = a,d$$

Analysis of Eqs. (2–4) shows that the liquid responds to a disturbance of the gyroscope by excitation of small-amplitude oscillations superposed on the state of rigid-body spin. The associated pressure fluctuations acting at the cavity walls produce a moment on the gyroscope. When this moment is substituted into the right side of Eq. (1), it is found that the motion of the gyroscope can grow without limit under certain adverse conditions that involve excitation of an eigenfrequency of the liquid.

Solution of the linear Eqs. (2–4) is relatively straightforward. Following Stewartson's procedure (separation of variables), we find the moment to be represented by

$$F = -i\Gamma\lambda_0\exp(i\Omega\tau t) \qquad (5)$$

where λ_0 is a constant and, per assumption (2) (relatively small liquid mass), many terms of Γ vanish, leaving

$$\Gamma(\tau) = -2\pi\rho c\xi\tau/(\tau + 1) \qquad (6)$$

where¶

$$\xi \equiv \sum_{j=0}^{\infty} C_j^2[a\Xi_1(\alpha k_j a) - d\,\Xi_1(\alpha k_j d)]; \quad C_j = \frac{-2}{ck_j^2} \qquad (7)$$

and

$$\Xi_\gamma(\alpha k_j r) = X_j J_\gamma(\alpha k_j r) + Z_j Y_\gamma(\alpha k_j r); \quad \gamma = 0,1 \qquad (8)$$

and

$$\alpha^2 = (1 + \tau)(3 - \tau)/(1 - \tau)^2 \qquad (9)$$

The constants X_j, Z_j are determined by the two relations

$$(1 - \tau)(\alpha k_j r)\Xi_0(\alpha k_j r) + (1 + \tau)\Xi_1(\alpha k_j r) = 2r\Omega^2\tau(1 - \tau)(\tau - 3) \qquad r = a,d \qquad (10)$$

so long as the determinant formed by the left-hand sides of these equations is not zero. The singular circumstance obtains for an infinite set of values of τ (eigenfrequencies of the liquid), which depend upon the cavity geometry and mode numbers n, j in the following way:

$$\tau_{nj} = f_n[(d/a)^2, c/a(2j + 1)] \qquad (11)$$

We can examine stability by substituting Eq. (5) into Eq. (1). The result submits to solution by the substitution

$$\lambda = \lambda_0 e^{i\Omega\tau t} \qquad (12)$$

and provides

$$T\tau^2 - L\tau + M_0/\Omega^2 = \Gamma/\Omega^2 \qquad (13)$$

In view of Eq. (12), we see that instability or stability of the system relates to whether or not τ, as given by Eq. (13), has a negative imaginary part. Hence, we wish to examine the nature of the roots of Eq. (13). With our assumptions, Eq. (13) becomes

$$T(\tau - \tau_{NU})(\tau - \tau_{PR}) = -2\pi\rho c\xi\tau/(\tau + 1)\Omega^2 \qquad (14)$$

where

$$\left.\begin{matrix}\tau_{NU}\\ \tau_{PR}\end{matrix}\right\} = \left(\frac{L}{2T}\right)[1 \pm (1 - \beta)^{1/2}] \qquad (15)$$

and $\beta = 4M_0T/(L\Omega)^2$. Study of Eq. (14) has shown that the right-hand side is negligible compared to the left except when a root τ is in the vicinity of any of the poles** of the former. If the right side is negligible, we get as roots, τ_{NU} and τ_{PR}. Since the gyroscope is stable as long as $\beta < 1$ in this conventional situation, we assume this always is the case. For roots near a pole, the right side is dominated by the individual singular term, and Eq. (14) becomes

$$T(\tau - \tau_{NU})(\tau - \tau_{PR}) = -\rho a^6 R_{nj}^2/c(\tau - \tau_{nj}) \qquad (16)$$

Here τ_{nj} is the eigenfrequency of the liquid corresponding to

‡ A comprehensive coverage of the behavior of rotating liquids is provided by Ref. 2.

§ The essential difference between our problem and Stewartson's is the boundary condition at $r' = d$. He has constant pressure on an internal boundary.

¶ This infinite sum must be considered judiciously[2]; experience indicates the treatment used here is satisfactory in practice.

** That is, except for a root of Eq. (14) near any of the τ_{nj}.

the specific pole in question, and R_{nj} is the residue at the pole. We now remark that a root of Eq. (16) has negative imaginary parts only if the root and τ_{nj} are near τ_{NU}. Within current approximations, this root can be evaluated by expanding the left side of Eq. (16) in a Taylor series about τ_{NU} and discarding all terms of second and higher order in τ to obtain:

$$T(\tau - \tau_{NU})(\tau - \tau_{nj}) + \rho a^6 R_{nj}^2/c(\tau_{NU} - \tau_{PR}) = 0 \quad (17)$$

The roots of Eq. (17) are

$$\tau_{1,2} = \tfrac{1}{2}(\tau_{NU} + \tau_{nj}) \pm \tfrac{1}{2}[(\tau_{NU} - \tau_{nj})^2 - \zeta]^{1/2} \quad (18)$$

where

$$\zeta \equiv 4\rho a^6 R_{nj}^2/cL(1 - \beta)^{1/2} \quad (19)$$

One of the roots has a negative imaginary part, and the gyroscope is predicted to be unstable, whenever

$$(\tau_{NU} - \tau_{nj})^2 < \zeta \quad (20)$$

otherwise the system is stable. When the system is unstable the nutational component of yaw grows as $\exp(\alpha_I \Omega t)$, where

$$\alpha_I = \tfrac{1}{2}[\zeta - (\tau_{NU} - \tau_{nj})^2]^{1/2} \quad (21)$$

Thus, for the inviscid case, the eigenfrequencies of the liquid depend on the cavity geometry through a relation of the form of Eq. (11); the liquid-gyroscope motion is stable except when a liquid eigenfrequency τ_{nj} is near the nutational frequency of the gyroscope, Eq. (20); in an unstable condition, the amplitude of the motion behaves as given by Eq. (21). To use these results, it is necessary to know the specific form of Eq. (11) and the values of the R_{nj}, both of which are provided by rather complicated functions, which we have evaluated[3] numerically for selected values of $(d/a)^2$ and $n = 1$ for τ_{nj} in the range 0–0.3 (Figs. 2 and 3).

Analysis of Viscous Effects

Wedemeyer[4] determined the effect of viscosity on the eigenfrequencies and then derived a new stability criterion for Stewartson's[1] problem. The latter step is straightforward once the new eigenfrequencies τ_{nj}^v are known. The assumptions and many of the results of Wedemeyer's work apply directly to our problem. Hence, we will derive only the new results required; specifically, we must account for viscosity in the presence of the rod-liquid interface. By so doing we will cover the most important features of Wedemeyer's approach. The liquid is assumed to be inviscid except in thin laminar boundary layers at the liquid-solid interfaces of the cavity. To begin the analysis, we use the results of the preceding section and represent the fluid velocity as small oscillatory motions superimposed upon a rigid body rotation about a vertical axis. Referred to the inertial frame $OXYZ$, the velocity is

$$\mathbf{V}(\mathbf{R},t) = \Omega \mathbf{i}_Z \times \mathbf{R} + \mathbf{u}(\mathbf{R},t) \quad (22)$$

while the equations of motion and of continuity are

$$\partial \mathbf{u}/\partial t + 2\Omega \mathbf{i}_Z \times \mathbf{u} + \Omega[\mathbf{s} \times \nabla(\mathbf{u}\cdot\mathbf{i}_Z) - \mathbf{i}_Z \times \nabla(\mathbf{u}\cdot\mathbf{s})] + \nabla(p^*/\rho) = \nu \nabla^2 \mathbf{u} \quad (23)$$

$$\nabla \cdot \mathbf{u} = 0 \quad (24)$$

Solution of Eqs. (23) and (24) for appropriate boundary conditions is found by considering $\mathbf{u}$ and p^* each to be composed of two parts: The inviscid contribution with $\mathbf{u}_0, p_0^*$ corresponds to the inviscid solution and satisfies Eqs. (23) and (24) for $\nu = 0$. The viscous contribution $\mathbf{u}_1, p_1^*$ satisfies relations derived from Eqs. (23) and (24) upon making the usual laminar boundary-layer approximations[5] for large Re. At the rod-liquid interface the appropriate equations referred

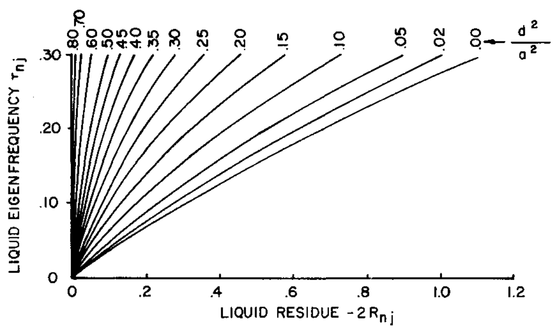

Fig. 3 Residues as a function of eigenfrequency and rod diameter for $n = 1$. See Ref. 3 for a more detailed tabulation.

to cylindrical polar coordinates are††

$$\partial v_1/\partial t + \Omega \partial v_1/\partial \phi = \nu \partial^2 v_1/\partial \eta^2 \quad (25)$$

$$\partial w_1/\partial t + \Omega \partial w_1/\partial \phi = \nu \partial^2 w_1/\partial \eta^2 \quad (26)$$

$$\partial v_1/\partial \phi + \partial(s w_1)/\partial Z = -\partial(s u_1)/\partial \eta \quad (27)$$

where $\eta = s - d$. Boundary conditions express the no-slip requirement at $s = d$ and are

$$\mathbf{u}_0 + \mathbf{u}_1 = \mathbf{0} \text{ at } \eta = 0 \quad (28)$$

Furthermore, we require

$$\mathbf{u}_1 = \mathbf{0} \text{ at } \eta = \infty \quad (29)$$

The inviscid solution shows that $\mathbf{u}_0 = \mathbf{u}_0^*(s,Z) \exp[i(\Omega \tau t - \phi)]$. If we assume the same functional form for $\mathbf{u}_1$, i.e., $\mathbf{u}_1 = \mathbf{u}_1^*(s,Z) \exp[i(\Omega \tau t - \phi)]$, substitute it into Eqs. (25) and (26) and solve the resulting differential equations subject to the boundary conditions (28) and (29), we find

$$v_1^* = -v_0(d) e^{-\eta/\delta d}, \; w_1^* = -w_0(d) e^{-\eta/\delta d} \quad (30)$$

where

$$\delta d/a = (1 + i)/[2(1 - \tau)Re]^{1/2} \quad (31)$$

The inviscid velocity components at $s = d$ are $v_0^*(d)$ and $w_0^*(d)$.

Next we must integrate Eq. (27). The notations $\mathbf{u}_0 = \mathbf{u}_0^+ e^{-\eta/\delta d}$ and $\mathbf{u}_1 = \mathbf{u}_1^+ e^{-\eta/\delta d}$ are introduced and used with Eqs. (30) and (31) to yield $v_1 = -v_0^+(d,\phi,Z,t)\, e^{-\eta/\delta d}$ and $w_1 = -w_0^+(d,\phi,Z,t)\, e^{-\eta/\delta d}$. Substituting these expressions into (27), integrating with respect to η, and applying Eq. (30), we get

$$s u_1 = -(\delta d)[\partial v_0^+/\partial \phi + \partial(s w_0^+)/\partial Z]_{s=d}\, e^{-\eta/\delta d} \quad (32)$$

Equation (32), evaluated at $\eta = 0$ in conjunction with the continuity equation for the inviscid solution (i.e., $\nabla \cdot \mathbf{u}_0 = 0$), leads to

$$[s u_0 + (\delta d)\partial(s u_0)/\partial s]_{s=d} = 0 \quad (33)$$

To an acceptable approximation Eq. (33) is

$$u_0(d) + (\partial u_0/\partial s)_{s=d}\delta d = u_0(d + \delta d) = 0 \quad (34)$$

Equation (34) embodies the effect of the boundary layer at the rod-liquid interface. It shows that solution of the viscous problem with $\mathbf{u} = \mathbf{0}$ at $s = d$ is provided by a corresponding inviscid solution with boundary condition $u_0 = 0$ at $s = d + \delta d$. The boundary layer effectively alters the dimension of the cavity. Parallel results apply for the effects of viscosity at $s = a$ and $Z = h \pm c$. Wedemeyer's earlier work shows that modified boundary conditions are

$$u_0(a - \delta a) = w_0[h \pm (c - \delta c)] = 0 \quad (35)$$

†† Equations (25–27) are linear, as is Eq. (24) when $\nu = 0$. Thus, superposition of the solutions $\mathbf{u}_0 + \mathbf{u}_1 = \mathbf{u}$ is permissible.

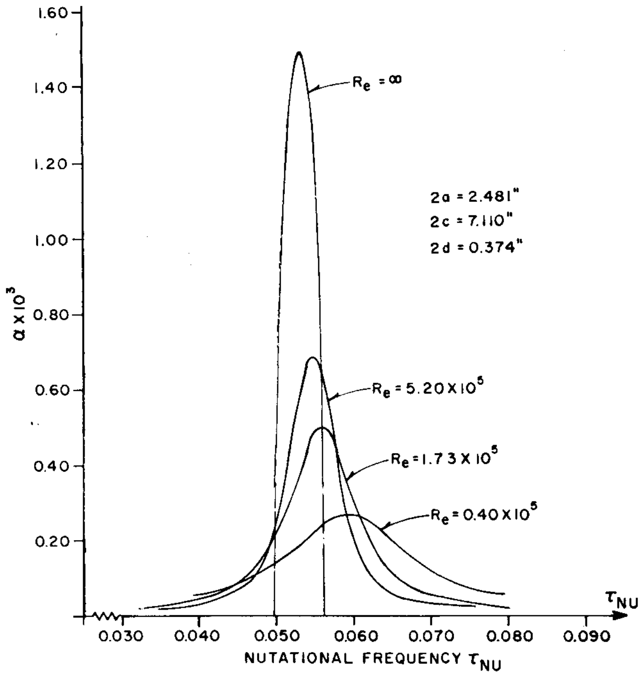

Fig. 4 Predicted exponential yaw growth rate vs τ_{NU}, Eq. (41) for several Reynolds numbers.

where $\delta a/a$ is given by the right side of Eq. (31) and

$$\delta c/a = [(1 - i)(3 - \tau)/(1 + \tau)^{1/2} - (1 + i)(1 + \tau)/(3 - \tau)^{1/2}]/(1 - \tau)2^{3/2}Re^{1/2} \quad (36)$$

A basic effect of these viscosity-modified boundary conditions is a corresponding modification of the eigenfrequencies, which are representable as

$$\tau_{nj}{}^{v}(a,d,c) = \tau_{nj}(a - \delta a, d + \delta d, c - \delta c) \quad (37)$$

where the τ_{nj} are those that would be given by the inviscid analysis when the modified boundary conditions are used.

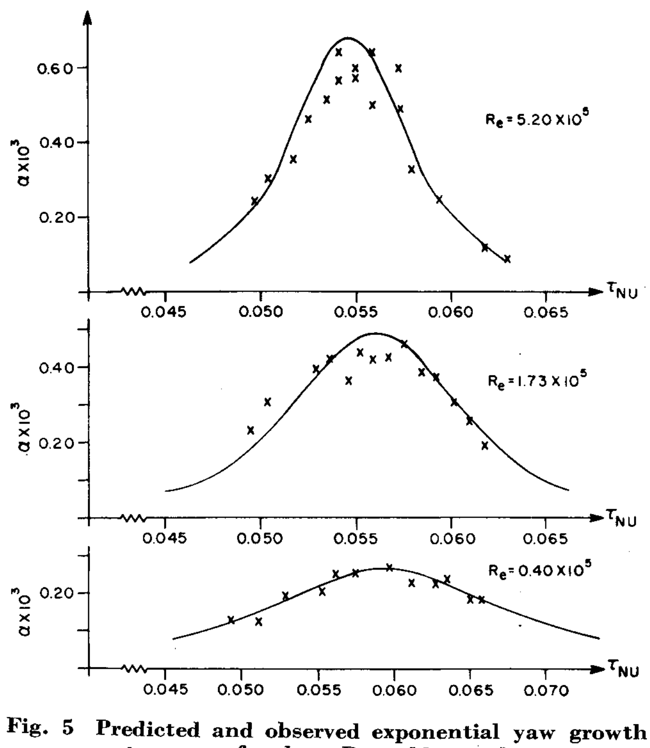

Fig. 5 Predicted and observed exponential yaw growth rates vs τ_{NU} for three Reynolds numbers.

Equations (34) and (35) apply so long as $|\delta a|/a \ll 1$, $|\delta c|/c \ll 1$, and $|\delta d|/d \ll 1$ [usually, this requires that $d^2\Omega/\nu \geq 0(10^4)$]. Consistent with this requirement, we can write Eq. (37) as

$$\tau_{nj}{}^{v}(a,d,c) = [1 - (\delta a\partial/\partial a - \delta d\partial/\partial d + \delta c\partial/\partial c)]\tau_{nj}(a,d,c) \quad (38)$$

Since δa, δd, δc are complex it is convenient to rewrite Eq. (38) as

$$\tau_{nj}{}^{v} = \tau_{nj} + \Delta\tau_{nj} = \tau_{nj}{}^{*} + i\delta_{nj} \quad (39)$$

where

$$\tau^{*}{}_{nj} = \tau_{nj} + \mathrm{Real}(\Delta\tau_{nj}),\ \delta_{nj} = \mathrm{Imag}(\Delta\tau_{nj})$$

Now we can determine $\tau_{nj}{}^{v}$ by using Eq. (38) and Figs. 2 and 3 (or related tables[3]). (Within current approximations, it is not necessary to account for changes in the residues R_{nj}.) Finally, we adopt the assumptions (except for zero viscosity) leading to Eq. (16) and apply the viscous correction to Eq. (16) simply by replacing τ_{nj} by $\tau_{nj}{}^{v}$ $(= \tau_{nj}{}^{*} + i\delta_{nj})$. Following arguments identical to those in the inviscid analysis we find that strongly unstable motion is predicted whenever $\tau_{nj}{}^{v}$ is near τ_{NU}; in this circumstance, the roots of the equation are:

$$\tau_{1,2} = \tfrac{1}{2}(\tau_{nj}{}^{*} + \tau_{NU}) \pm \{[(\tau_{nj}{}^{*} - \tau_{NU})/2]^2 - (\delta_{nj}/2)^2 + i\delta_{nj}(\tau_{nj}{}^{*} - \tau_{NU})/2 - \zeta^2/4\}^{1/2} + i\delta_{nj}/2 \quad (40)$$

where ζ is given by Eq. (19). The yaw growth rate of the gyroscope is given by the negative imaginary part of Eq. (40),

$$\alpha_v = \mathrm{Pos.\ Real}(\{\zeta/4 - [(\tau_{nj}{}^{*} - \tau_{NU})/2 + i\delta_{nj}/2]^2\}^{1/2}) - \delta_{nj}/2 \quad (41)$$

where the amplitude of the gyroscope's motion behaves as $\lambda \propto \exp(\alpha\Omega_v t)$. Maximum growth rate occurs when $\tau_{nj}{}^{*} = \tau_{NU}$.

Experiments and Conclusions

An experimental check has been made with a gyroscope[6] having a rotor designed to accommodate liquid-filled cavities and instrumentation for recording yaw amplitude vs time. The records are easily reduced to provide τ_{NU} and α_v of Eq. (41). It is possible to control τ_{NU} by adding brass rings to the rotor; the range available is $0.020 \leq \tau_{NU} \leq 0.140$ in increments of ~0.001. Experiments were conducted for a set of three Re's (Fig. 4) obtained by use of silicon oils of various viscosities, for a cavity with $2a = 2.481$ in., $2d = 0.374$ in., and $2c = 7.110$ in., and for a gyroscope with $L = 65.04$ lb-in.[2] (nominal, $\beta = 0$), and $\Omega = 5000$ rpm. Figures 4 and 5 display the results. Figure 4 also shows the curve for α vs τ_{NU} for $\nu = 0$ ($Re = \infty$). Comparison of this inviscid theory curve with the data points in Fig. 5 show significant differences at each Re tested, but in each case, very good agreement is seen with viscous correction. Thus, the analysis is adequate so long as the amplitude of nutation is small.

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